The double scaling limit method in the Toda hierarchy

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Abstract
Critical points of semiclassical expansions of solutions to the dispersionful Toda hierarchy are considered and a double scaling limit method of regularization is formulated. The analogues of the critical points characterized by the strong conditions in the Hermitian matrix model are analysed and the property of doubling of equations is proved. A wide family of sets of critical points is introduced and the corresponding double scaling limit expansions are discussed.

PACS number: 02.30.Ik.

1. Introduction

In a recent work [1] we considered the large N expansion of the Hermitian matrix model

$$Z_N(t) = \int \mathrm{d}H \exp\left(\frac{N}{g}\mathrm{tr}\left(\sum_{k\geqslant 1} t_k H^k\right)\right), \qquad H = H^\dagger, \qquad (1)$$

from the point of view of the *dispersionful* Toda hierarchy [2]

$$\epsilon\frac{\partial\mathcal{L}}{\partial t_j} = [(\mathcal{L}^j)_+, \mathcal{L}], \qquad \mathcal{L} = \Lambda + u + v\Lambda^{-1}, \qquad \Lambda := \exp(\epsilon\partial_x), \qquad \epsilon := g/N, \quad (2)$$

where $()_+$ denotes the polynomial part in Λ. If the equilibrium measure in the large N limit is supported on a single interval (one-cut case), then the partition function becomes a tau-function $\tau(x, t)$ of this hierarchy at $x = g$. The corresponding solution of (2) is characterized by two string equations which can be solved in terms of a *semiclassical* expansion

$$u = \sum_{k\geqslant 0}\epsilon^k u^{(k)}(x, t), \qquad v = \sum_{k\geqslant 0}\epsilon^{2k} v^{(2k)}(x, t). \qquad (3)$$

The coefficients $(u^{(k)}, v^{(2k)})$ can be expressed as rational functions of the leading terms $(u^{(0)}, v^{(0)})$ and their derivatives with respect to x.

A method to determine expansions (3) was provided in [1]. It uses two functions

$$\mathbb{R} := \sum_{k\geqslant 0}\frac{R_k(u, v)}{z^k}, \qquad \mathbb{T} := \sum_{k\geqslant 0}\frac{T_k(u, v)}{z^k}, \qquad R_0 = T_0 = 1,$$

which are related to the resolvent $\mathcal{R} := (z - \mathcal{L})^{-1}$ of the Lax operator $\mathcal{L}$ through the equation

$$\left(z - \frac{2\mathbb{R}}{(1 + \mathbb{T})} \Lambda \right) \mathcal{R}_+ = \mathbb{R},$$

so that they generate the trace densities $\mathrm{Res}(\mathcal{R}\Lambda^{-j})$ for $j \geqslant 0$, where $\mathrm{Res}\left(\sum c_k \Lambda^k \right) := c_0$. These functions are determined by the equations

$$\begin{cases} \mathbb{T} + \mathbb{T}_{[-1]} + \dfrac{2}{z}(u - z)\mathbb{R} = 0, \\ \mathbb{T}^2 - \dfrac{4}{z^2} v_{[1]} \mathbb{R}\mathbb{R}_{[1]} = 1. \end{cases} \tag{4}$$

where we denote

$$F_{[r]}(x) := F(x + r\epsilon).$$

As was shown in [1] the string equations for (u, v) reduce to the system

$$\begin{cases} \displaystyle\oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}z} V_z(z, \boldsymbol{t})\mathbb{R}(z, u, v) = 0, \\ \displaystyle\oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \boldsymbol{t})\mathbb{T}_{[-1]}(z, u, v) = -2x, \end{cases} \tag{5}$$

where $V(z, \boldsymbol{t}) := \sum_{j=1}^\infty t_j z^j$, and γ is a large positively oriented closed path. For $\epsilon = 0$ system (5) reduces to a pair of hodograph-type equations

$$f_v(\boldsymbol{t}, u^{(0)}, v^{(0)}) = 0, \qquad f_u(\boldsymbol{t}, u^{(0)}, v^{(0)}) = 2x, \tag{6}$$

where

$$f(\boldsymbol{t}, u, v) := \frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathrm{d}z\, V_z(z, \boldsymbol{t})((z - u)^2 - 4v)^{\frac{1}{2}} = -2vt_1 - 4uvt_2 + \cdots. \tag{7}$$

These equations determine $(u^{(0)}, v^{(0)})$ and represent the planar limit contribution to the partition function of the Hermitian model [6]. However, near critical points of (6) the functions $(u^{(0)}, v^{(0)})$ are multi-valued and have singular x derivatives (*gradient catastrophe*), so that the semiclassical expansion (3) breaks down. In terms of the Hermitian matrix model this situation corresponds to the critical points of the large N expansion of $Z_N(\boldsymbol{t})$ and, as it is well known [3–12], a *double scaling limit* method of regularization is available. This method leads to an important nonperturbative approach to two-dimensional quantum gravity.

Recent research [13–16] shows that the double scaling limit expansions are also relevant in the asymptotic analysis of solutions of dispersionful integrable systems. They supply good approximations in the transition regions from the semiclassical to the oscillatory regimes after the time of gradient catastrophe. These applications motivate the interest in reconsidering the double scaling limit method from the point of view of the theory of string equations for integrable systems of dispersionful type. The formulation of the double scaling limit within the theory of the dispersionful Toda hierarchy was already addressed in [1], but only the case corresponding to $u = 0$ and even potentials $V(-z) = V(z)$ was analysed. The purpose of the present paper is to analyse the general case that exhibits a much richer structure of critical behaviours (see [10–12] for its relevance in the Hermitian matrix model of random surfaces and quantum gravity).

According to the strategy of the double scaling method we look at systems (4) and (5) as singular perturbation problems in the small parameter ϵ and consider the semiclassical expansions near critical points as *outer* expansions of the solutions of (4) and (5). Consequently the problem consists in characterizing appropriate *inner* expansions

$$x = x_c + \bar{\epsilon}^{2n}\bar{x}, \qquad u = u_c + \sum_{j \geqslant 2} \bar{\epsilon}^j \bar{u}^{(j)}, \qquad v = v_c + \sum_{j \geqslant 2} \bar{\epsilon}^j \bar{v}^{(j)}, \qquad \epsilon = \bar{\epsilon}^{2n+1} \tag{8}$$

with coefficients depending on stretched variables $(\overline{x}, \overline{t})$. This type of expansions were determined in the theory of random matrix models through several methods of analysis of discrete string equations for orthogonal polynomials in the large N limit. The present work provides an alternative scheme based on the characterization of solutions

$$\mathbb{R} = \sum_{j \geqslant 0} \overline{\epsilon}^j \overline{R}^{(j)}, \qquad \mathbb{T} = \sum_{j \geqslant 0} \overline{\epsilon}^j \overline{T}^{(j)},$$

of the resolvent trace equations (4) of the dispersionful Toda hierarchy. This scheme establishes the existence of series (8) to all orders in $\overline{\epsilon}$ and leads to the characterization of the subleading coefficients $(\overline{u}^{(2)}, \overline{v}^{(2)})$ in terms of ordinary differential equations involving the Gel'fand–Dikii polynomials of the resolvent trace expansion of the Schrödinger operator.

We analyse two different classes of critical points. Firstly, we consider the sets $\mathcal{C}_n$ of critical points which underlie the Hermitian matrix model of quantum gravity [10, 11] and we determine inner expansions (8) which satisfy the *doubling property*

$$c_{\pm} G_{n,\pm}(u_{\pm}) = \overline{x}, \qquad u_{\pm} := v^{(2)} \pm \sqrt{v_c} u^{(2)},$$

with $G_{n,\pm}$ being the Gel'fand–Dikii polynomials. Secondly, we characterize a wide family of sets $\mathcal{C}_{lmn}$ of critical points where $l \geqslant 1, m = 1, \ldots, l$ and $n \geqslant 2$, describe the branching behaviour of $(u^{(0)}, v^{(0)})$ near the critical points. Then we develop the double scaling limit method for the case $n = 2$ with stretched variables

$$x = x_c + \overline{\epsilon}^4 \overline{x}, \qquad t_j = t_{c,j} + \overline{\epsilon}^4 \overline{t}_j, \qquad j \geqslant 1, \qquad \overline{\epsilon} = \epsilon^{1/5}.$$

For $\mathcal{C}_{112}$ we find that the subleading coefficients satisfy

$$\overline{v}^{(2)} = \pm \sqrt{v_c} \overline{u}^{(2)},$$

with $\overline{v}^{(2)}$ being determined by the Painlevé I equation. For the other cases $\mathcal{C}_{lm2}, l \geqslant 2$, we find again the doubling property. Indeed, the functions

$$u_{\pm} := \overline{v}^{(2)} \pm \sqrt{v_c} \overline{u}^{(2)},$$

turn out to be determined by a pair of decoupled Painlevé I equations.

The double scaling limit expansions at critical points of the types $\mathcal{C}_{lmn}$ can be applied to the asymptotic analysis of solutions of the dispersionful Toda hierarchy. As an illustrative example we end our work by applying our results to regularize some critical processes of ideal models of Hele–Shaw flows [18–20] based on the Toda hierarchy.

Our scheme can be formulated with other scaling ansatz for stretched variables like those of Galilean type used in [13–16]. Furthermore, it can be applied to the study of the double scaling limit of the normal matrix model which is related to another solution of the two-dimensional dispersionful Toda hierarchy [21–23] determined by a pair of string equations. We also expect the methods of this paper to be applicable to dispersionful versions of the multi-component KP hierarchies which arise in random matrix theory and Dyson Brownian motions [26].

The layout of the paper is as follows: in section 2 we show how inner expansions of the generating functions $\mathbb{R}$ and $\mathbb{T}$ can be determined from the resolvent trace equations (4). In section 3 we characterize the double scaling limit expansions at critical points satisfying strong conditions. Finally, we devote section 4 to introduce families $\mathcal{C}_{lmn}$ of critical points and to develop the corresponding double scaling limit method.

2. Inner expansions of the resolvent functions

Let us consider the hodograph system (6)

$$f_v = 0, \qquad f_u = 2x. \tag{9}$$

Note that the function f defined in (7) satisfies

$$f_{uu} - v f_{vv} = 0. \tag{10}$$

The hodograph system determines a solution of the dispersionless Toda hierarchy provided that the implicit function theorem applies for obtaining $(u(x, t), v(x, t))$ from (9). For example, (9) implies

$$u_{t_1} = v_x, \qquad v_{t_1} = v u_x,$$

so that

$$v_{xx} - (\log v)_{t_1 t_1} = 0,$$

which is the hyperbolic version of the long-wave limit of the Toda lattice equation. It is also easy to see that the solutions of (9) verify

$$u_{t_2} = 2(u u_{t_1} + v_{t_1}), \qquad v_{t_2} = 2(v u)_{t_1},$$

which is the dispersionless limit of the focusing nonlinear Schrödinger equation [16].

The set of critical points (x_c, t_c, u_c, v_c) of (9) is characterized by the system

$$f_v = 0, \qquad f_u = 2x, \qquad f_{uv}^2 - f_{uu} f_{vv} = 0. \tag{11}$$

Our main aim in this work is to characterize inner expansions for solutions (u, v) of systems (4) and (5) near critical points of (9).

2.1. Symmetric variables

Let us start by analysing the resolvent trace system (4). It is form invariant under the change of variables

$$x = x_c + \bar{\epsilon}^{2n} \bar{x}, \qquad \epsilon = \bar{\epsilon}^{2n+1}, \qquad n \geqslant 2. \tag{12}$$

Indeed since $\epsilon \partial_x = \bar{\epsilon} \partial_{\bar{x}}$ we can rewrite (4) as

$$\begin{cases} \mathbb{T} + \mathbb{T}_{[-\bar{1}]} + \dfrac{2}{z}(u - z)\mathbb{R} = 0, \\ \mathbb{T}^2 - \dfrac{4}{z^2} v_{[\bar{1}]} \mathbb{R} \mathbb{R}_{[\bar{1}]} = 1, \end{cases} \tag{13}$$

where we denote

$$F_{[\bar{r}]}(\bar{x}) := F(\bar{x} + r\bar{\epsilon}).$$

We will prove that if (u, v) are expansions of the form

$$u = u_c + \sum_{j \geqslant 2} \bar{\epsilon}^j \bar{u}^{(j)}, \qquad v = v_c + \sum_{j \geqslant 2} \bar{\epsilon}^j \bar{v}^{(j)}, \tag{14}$$

then the solution of (13) can be expressed as

$$\mathbb{T} = \sum_{j \geqslant 0} \bar{\epsilon}^j \overline{T}^{(j)}, \qquad \mathbb{R} = \sum_{j \geqslant 0} \bar{\epsilon}^j \overline{R}^{(j)}, \tag{15}$$

where the coefficients $(\overline{T}^{(j)}, \overline{R}^{(j)})$ are differential polynomials in the coefficients of the expansions of (u, v).

An important property of the combined system (4) and (5) is its invariance under the transformation

$$\begin{aligned} \widehat{u}(\epsilon, \bar{x}) &:= u(-\bar{\epsilon}, \bar{x} + \bar{\epsilon}), & \widehat{v}(\epsilon, \bar{x}) &:= v(-\bar{\epsilon}, \bar{x}), \\ \widehat{\mathbb{T}}(\bar{\epsilon}, z, \bar{x}) &:= \mathbb{T}(-\bar{\epsilon}, z, \bar{x} + 2\bar{\epsilon}), & \widehat{\mathbb{R}}(\bar{\epsilon}, z, \bar{x}) &:= \mathbb{R}(-\bar{\epsilon}, z, \bar{x} + \bar{\epsilon}). \end{aligned} \tag{16}$$

Thus it is convenient to introduce the *symmetric* variables

$$w(\overline{\epsilon}) := u\left(\overline{\epsilon}, \overline{x} - \frac{\overline{\epsilon}}{2}\right) = u_{[\overline{-1/2}]}(\overline{\epsilon}), \tag{17}$$

$$\mathbb{U}(\overline{\epsilon}) := \mathbb{T}(\overline{\epsilon}, z, \overline{x} - \overline{\epsilon}) = \mathbb{T}_{[\overline{-1}]}(\overline{\epsilon}) \qquad \mathbb{V}(\overline{\epsilon}, z, \overline{x}) := \mathbb{R}\left(\overline{\epsilon}, z, \overline{x} - \frac{\overline{\epsilon}}{2}\right) = R_{[\overline{-1/2}]}(\overline{\epsilon}), \tag{18}$$

which transform under (16) in the simple form

$$\widehat{w}(\overline{\epsilon}) = w(-\overline{\epsilon}), \qquad \widehat{\mathbb{U}}(\overline{\epsilon}) := \mathbb{U}(-\overline{\epsilon}), \qquad \widehat{\mathbb{V}}(\overline{\epsilon}) = \mathbb{V}(-\overline{\epsilon}). \tag{19}$$

In terms of $(w, v, \mathbb{U}, \mathbb{V})$ system (13) reads as

$$\begin{cases} \mathbb{U}_{[\overline{1}]} + \mathbb{U} + \dfrac{2}{z}(w_{[\overline{1/2}]} - z)\mathbb{V}_{[\overline{1/2}]} = 0, \\[2mm] \mathbb{U}^2 - \dfrac{4}{z^2}v\mathbb{V}_{[\overline{-1/2}]}\mathbb{V}_{[\overline{1/2}]} = 1. \end{cases} \tag{20}$$

Moreover, from (20) one easily deduces the following linear equation for $\mathbb{U}$:

$$(z - w_{[\overline{-1/2}]})(z - w_{[\overline{1/2}]})(z - w_{[\overline{3/2}]})(\mathbb{U}_{[\overline{1}]} - \mathbb{U}) = v_{[\overline{1}]}(z - w_{[\overline{-1/2}]})(\mathbb{U}_{[\overline{2}]} + \mathbb{U}_{[\overline{1}]})$$
$$- v(z - w_{[\overline{3/2}]}))(\mathbb{U}_{[\overline{-1}]} + \mathbb{U}). \tag{21}$$

It can be expressed as

$$\left(\sum_{k \geqslant 1} \overline{\epsilon}^k \mathcal{J}_k\right)\mathbb{U} = 0, \tag{22}$$

where $\mathcal{J}_k$ are linear differential operators depending on the coefficients of the expansions of (w, v) of the form

$$\mathcal{J}_k = (z - u_c)r_c\mathcal{J}_{k,0} + (z - u_c)\mathcal{J}_{k,1} + r_c\mathcal{J}_{k,2} + \mathcal{J}_{k,3},$$

where

$$r_c = r_c(z) := (z - u_c)^2 - 4v_c,$$

and $\mathcal{J}_{k,i}$ are z independent. For example, the first few are

$$\mathcal{J}_1 = (z - u_c)r_c\partial_{\overline{x}}, \qquad \mathcal{J}_2 = \frac{1}{2}(z - u_c)r_c\partial_{\overline{x}}^2,$$
$$\mathcal{J}_3 = \frac{z - u_c}{6}r_c\partial_{\overline{x}}^3 - 3r_c u^{(2)}\partial_{\overline{x}} - (z - u_c)\left(v_c\partial_{\overline{x}}^3 + 4v^{(2)}\partial_{\overline{x}} + 2v_{\overline{x}}^{(2)}\right) - 4v_c\left(2u^{(2)}\partial_{\overline{x}} + u_{\overline{x}}^{(2)}\right). \tag{23}$$

We will now use both (20) and (21) to determine the expansions of $\mathbb{U}$ and $\mathbb{V}$.

If we substitute into (20) the expansions

$$w = u_c + \sum_{j \geqslant 2}\overline{\epsilon}^j\overline{w}^{(j)}, \qquad v = v_c + \sum_{j \geqslant 2}\overline{\epsilon}^j\overline{v}^{(j)},$$
$$\mathbb{V} = \sum_{j \geqslant 0}\overline{\epsilon}^j\overline{V}^{(j)}, \qquad \mathbb{U} = \sum_{j \geqslant 0}\overline{\epsilon}^j\overline{U}^{(j)},$$

then by identifying the coefficients of the powers of $\overline{\epsilon}$ we find

$$\frac{\overline{V}^{(0)}}{z} = \frac{1}{r_c^{1/2}}, \qquad \overline{U}^{(0)} = \frac{z - u_c}{r_c^{1/2}}, \qquad \frac{\overline{V}^{(1)}}{z} = \overline{U}^{(1)} = 0,$$

and the recurrence system ($j \geqslant 2$)

$$\overline{U}^{(j)} - (z - u_c)\frac{\overline{V}^{(j)}}{z} = -\frac{1}{2}\sum_{k+i=j,k\geqslant 1}\frac{1}{k!}\partial_{\overline{x}}^k\overline{U}^{(i)} + (z - u_c)\sum_{k+i=j,k\geqslant 1}\frac{1}{2^k(k)!}\partial_{\overline{x}}^k\frac{\overline{V}^{(i)}}{z}$$

$$-\sum_{\substack{i_1+i_2+k_1+k_2=j \\ i_1\geqslant 2}}\frac{1}{2^{k_1+k_2}k_1!k_2!}(\partial_{\overline{x}}^{k_1}\overline{w}^{(i_1)})\left(\partial_{\overline{x}}^{k_2}\frac{\overline{V}^{(i_2)}}{z}\right),$$

$$\overline{U}^{(0)}\overline{U}^{(j)} - 4v_c\frac{\overline{V}^{(0)}}{z}\frac{\overline{V}^{(j)}}{z} = -\frac{1}{2}\sum_{i+k=j,1\leqslant i,k\leqslant j-1}\overline{U}^{(i)}\overline{U}^{(k)}$$

$$+\sum_{\substack{ci_1+i_2+k_1+k_2=j \\ 1\leqslant i_1,i_2\leqslant j-1,k_1+k_2 \text{ even}}}\frac{(-1)^{k_1}}{2^{k_1+k_2-1}k_1!k_2!}v_c\left(\partial_{\overline{x}}^{k_1}\frac{\overline{V}^{(i_1)}}{z}\right)\left(\partial_{\overline{x}}^{k_2}\frac{\overline{V}^{(i_2)}}{z}\right)$$

$$+\sum_{\substack{i_1+i_2+k_1+k_2+l=j \\ l\geqslant 2,k_1+k_2 \text{ even}}}\frac{(-1)^{k_1}}{2^{k_1+k_2}k_1!k_2!}\overline{v}^{(l)}\left(\partial_{\overline{x}}^{k_1}\frac{\overline{V}^{(i_1)}}{z}\right)\left(\partial_{\overline{x}}^{k_2}\frac{\overline{V}^{(i_2)}}{z}\right).$$

These equations are linear with respect to $\overline{U}^{(j)}$ and $\frac{\overline{V}^{(j)}}{z}$. Furthermore the determinant of their corresponding coefficients is $r_c^{1/2}$.

Thus one obtains

$$\overline{U}^{(2)} = \frac{1}{r_c^{\frac{3}{2}}}[4v_c\overline{w}^{(2)} + 2(z - u_c)\overline{v}^{(2)}], \qquad \frac{\overline{V}^{(2)}}{z} = \frac{1}{r_c^{\frac{3}{2}}}[2\overline{v}^{(2)} + (z - u_c)\overline{w}^{(2)}],$$

$$\overline{U}^{(3)} = \frac{1}{r_c^{\frac{3}{2}}}[4v_c\overline{w}^{(3)} + 2(z - u_c)\overline{v}^{(3)}], \qquad \frac{\overline{V}^{(3)}}{z} = \frac{1}{r_c^{\frac{3}{2}}}[2\overline{v}^{(3)} + (z - u_c)\overline{w}^{(3)}],$$

$$\overline{U}^{(4)} = \frac{1}{r_c^{\frac{3}{2}}}\left[4v_c\overline{w}^{(4)} + \frac{v_c}{2}\overline{w}_{\overline{xx}}^{(2)} + 4\overline{v}^{(2)}\overline{w}^{(2)} + 2(z - u_c)\overline{v}^{(4)}\right]$$

$$+\frac{1}{r_c^{\frac{5}{2}}}\left[4v_c^2\overline{w}_{\overline{xx}}^{(2)} + 24v_c\overline{v}^{(2)}\overline{w}^{(2)} + (z - u_c)\left(2v_c\overline{v}_{\overline{xx}}^{(2)} + 6\overline{v}^{(2)^2} + 6v_c\overline{w}^{(2)^2}\right)\right],$$

$$\frac{\overline{V}^{(4)}}{z} = \frac{1}{r_c^{\frac{3}{2}}}\left[2\overline{v}^{(4)} + \frac{1}{4}\overline{v}_{\overline{xx}}^{(2)} + \overline{w}^{(2)^2} + (z - u_c)\overline{w}^{(4)}\right]$$

$$+\frac{1}{r_c^{\frac{5}{2}}}\left[2v_c\overline{v}_{\overline{xx}}^{(2)} + 6v_c\overline{w}^{(2)^2} + 6\overline{v}^{(2)^2} + (z - u_c)\left(v_c\overline{w}_{\overline{xx}}^{(2)} + 6\overline{v}^{(2)}\overline{w}^{(2)}\right)\right],$$

and by using induction one proves that the coefficients of the expansions of $\mathbb{U}$ and $\mathbb{V}$ are of the form

$$\overline{U}^{(2j)} = \sum_{l=1}^{j}\frac{\alpha_l^{(2j)}(z - u_c) + \beta_l^{(2j)}}{r_c^{l+\frac{1}{2}}}, \;\; j \geqslant 3; \;\; \overline{U}^{(2j+1)} = \sum_{l=1}^{j}\frac{\alpha_l^{(2j+1)}(z - u_c) + \beta_l^{(2j+1)}}{r_c^{l+\frac{1}{2}}}, \;\; j \geqslant 2;$$

$$\frac{\overline{V}^{(2j)}}{z} = \sum_{l=1}^{j}\frac{\gamma_{2jl}(z - u_c) + \eta_{2jl}}{r_c^{l+\frac{1}{2}}}, \;\; j \geqslant 3; \;\; \frac{\overline{V}^{(2j+1)}}{z} = \sum_{l=1}^{j}\frac{\gamma_l^{(2j+1)}(z - u_c) + \eta_l^{(2j+1)}}{r_c^{l+\frac{1}{2}}}, \;\; j \geqslant 2.$$

$$(24)$$

Moreover by substituting (24) into the linear equation (21) and taking into account (23) it can be proved by induction that the functions $\alpha_l^{(i)}$, $\beta_l^{(i)}$, $\gamma_l^{(i)}$ and $\eta_l^{(i)}$ are differential polynomials in $(\overline{v}^{(2)}, \dots, \overline{v}^{(i-2l+2)}, \overline{w}^{(2)}, \dots, \overline{w}^{(i-2l+2)})$.

2.2. Invariant solutions

The solution $(\mathbb{U}, \mathbb{V})$ of system (20) is uniquely determined by (w, v). Hence if we assume that (w, v) are even functions of $\overline{\epsilon}$, then as a consequence of the invariance of (20) under transformation (16) we deduce that $(\mathbb{U}, \mathbb{V})$ are even functions of $\overline{\epsilon}$ too. In this way we have

$$w = u_c + \sum_{j \geqslant 1} \overline{\epsilon}^{2j} \overline{w}^{(2j)}, \qquad v = v_c + \sum_{j \geqslant 1} \overline{\epsilon}^{2j} \overline{v}^{(2j)},$$

$$\mathbb{U} = \sum_{j \geqslant 0} \overline{\epsilon}^{2j} \overline{U}^{(2j)}, \qquad \mathbb{V} = \sum_{j \geqslant 0} \overline{\epsilon}^{2j} \overline{V}^{(2j)}. \tag{25}$$

These solutions will be henceforth called *invariant solutions* of the resolvent trace equations.

As we have seen in the above subsection the coefficients $\overline{U}^{(2j)}$ and $\overline{V}^{(2j)}$ can be expanded in powers of r_c with leading terms

$$\overline{U}^{(2j)} = \frac{(z - u_c)\alpha_j^{(2j)} + \beta_j^{(2j)}}{r_c^{j+\frac{1}{2}}} + \mathcal{O}\left(\frac{1}{r_c^{j-\frac{1}{2}}}\right),$$

$$\frac{\overline{V}^{(2j)}}{z} = \frac{(z - u_c)\gamma_j^{(2j)} + \eta_j^{(2j)}}{r_c^{j+\frac{1}{2}}} + \mathcal{O}\left(\frac{1}{r_c^{j-\frac{1}{2}}}\right). \tag{26}$$

Furthermore, by identifying the coefficient of $\overline{\epsilon}^{2j}$ in the first equation of system (20) and by taking into account that $(z - u_c)^2 = r_c + 4v_c$ it follows that

$$\gamma_j^{(2j)} = \frac{\beta_j^{(2j)}}{4v_c}, \qquad \eta_j^{(2j)} = \alpha_j^{(2j)}. \tag{27}$$

Now we prove the connection between these coefficients and the Gel'fand–Dikii differential polynomials of the KdV theory.

Theorem 1. *The functions*

$$G_{j,\pm} := \frac{\beta_j^{(2j)}}{2v_c} \pm \frac{\alpha_j^{(2j)}}{\sqrt{v_c}}, \tag{28}$$

are the Gel'fand–Dikii differential polynomials in $u_\pm := v^{(2)} \pm \sqrt{v_c} u^{(2)}$, *respectively.*

Proof. By identifying the coefficient of $\overline{\epsilon}^{2j+1}$ in (22) we get

$$\sum_{k+2l=2j+1} \mathcal{J}_k \overline{U}^{(2l)} = 0. \tag{29}$$

From (23) and by taking into account that $(z - u_c)^2 = r_c + 4v_c$ it is clear that only the terms $\mathcal{J}_1 \overline{U}^{(2j)}$ and $\mathcal{J}_3 \overline{U}^{(2j-2)}$ contribute to the coefficient of $\frac{1}{r_c^{j-\frac{1}{2}}}$ in (29). Thus we get the recursion relations

$$\partial_{\overline{x}} \alpha_j^{(2j)} = \left(v_c \partial_{\overline{x}}^3 + 4\overline{v}^{(2)} \partial_{\overline{x}} + 2\overline{v}_{\overline{x}}^{(2)}\right) \alpha_{j-1}^{(2(j-1))} + \left(2\overline{u}^{(2)} \partial_{\overline{x}} + \overline{u}_{\overline{x}}^{(2)}\right) \beta_{j-1}^{(2(j-1))},$$

$$\partial_{\overline{x}} \beta_j^{(2j)} = \left(v_c \partial_{\overline{x}}^3 + 4\overline{v}^{(2)} \partial_{\overline{x}} + 2\overline{v}_{\overline{x}}^{(2)}\right) \beta_{j-1}^{(2(j-1))} + 4v_c\left(2\overline{u}^{(2)} \partial_{\overline{x}} + \overline{u}_{\overline{x}}^{(2)}\right) \alpha_{j-1}^{(2(j-1))}, \tag{30}$$

which lead at once to the well-known third-order differential equation for the Gel'fand–Dikii differential polynomials

$$\partial_{\overline{x}} G_{j,\pm} = \left(v_c \partial_{\overline{x}}^3 + 4u_\pm \partial_{\overline{x}} + 2u_{\pm,\overline{x}} \right) G_{j-1,\pm}. \tag{31}$$

$\square$

From (28) we have that $G_{1,\pm} = \pm\frac{2}{\sqrt{v_c}} u_\pm$ and by using (31) we find

$$G_{2,\pm} = \pm\frac{2}{\sqrt{v_c}} \left(v_c u_{\pm,\overline{xx}} + 3u_\pm^2 \right),$$

$$G_{3,\pm} = \pm\frac{2}{\sqrt{v_c}} \left(v_c^2 u_{\pm,\overline{xxxx}} + 10v_c u_{\pm,\overline{xx}} u_\pm + 5v_c u_{\pm,\overline{x}}^2 + 10u_\pm^3 \right),$$

$$G_{4,\pm} = \pm\frac{2}{\sqrt{v_c}} \left(v_c^3 u_{\pm,\overline{xxxxxx}} + 14v_c^2 u_\pm u_{\pm,\overline{xxxx}} + 28v_c^2 u_{\pm,\overline{xxx}} u_{\pm,\overline{x}} + 21v_c^2 u_{\pm,\overline{xx}}^2 \right.$$
$$\left. + 70v_c u_\pm^2 u_{\pm,\overline{xx}} + 70v_c u_\pm u_{\pm,\overline{x}}^2 + 35u_\pm^4 \right).$$

3. Strong conditions for critical points and the doubling property

In what follows the notation $F^{(c)}$ will represent the value of a function F at a critical point $(x_c, \boldsymbol{t}_c, u_c, v_c)$ of (9). We will also suppose that $v_c \neq 0$. The following sets of critical points were considered in the applications of the Hermitian matrix model to quantum gravity [10–12].

Definition 1. *Given $n \geqslant 2$ we denote by $\mathcal{C}_n$ the set of critical points of the hodograph system (9) which satisfy the (strong) conditions*

$$\left(\partial_v^k f \right)^{(c)} = \left(\partial_u \partial_v^l f \right)^{(c)} = 0, \qquad 1 \leqslant k \leqslant n, \qquad 1 \leqslant l \leqslant n-1, \tag{32}$$

and such that $\left(\left(\partial_v^{n+1} f \right)^{(c)}, \left(\partial_u \partial_v^n f \right)^{(c)} \right) \neq (0, 0)$.

Due to (10) it is clear that $\mathcal{C}_n$ is also determined by the condition that all derivatives $\left(\partial_u^k \partial_v^l f \right)^{(c)}$ with $(k, l) \neq (1, 0)$ up to order n vanish and such that at least one $(n+1)$-order derivative is different from zero.

An alternative characterization of $\mathcal{C}_n$ can be formulated in terms of the integrals

$$I_k(\boldsymbol{t}_c, u_c, v_c) := \frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathrm{d}z \frac{V_z(z, \boldsymbol{t}_c)}{r_c^{k+1/2}}, \qquad J_k(\boldsymbol{t}_c, u_c, v_c) := \frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathrm{d}z (z - u_c) \frac{V_z(z, \boldsymbol{t}_c)}{r_c^{k+1/2}}. \tag{33}$$

Indeed, I_k and J_k are proportional to $\left(\partial_v^{k+1} f \right)^{(c)}$ and $\left(\partial_u \partial_v^k f \right)^{(c)}$, respectively. Hence it follows that

Lemma 1. $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_n$ *if and only if*

$$I_0 = 0, \quad J_0 = -2x_c, \quad I_k = J_k = 0, \quad for \quad 1 \leqslant k \leqslant n-1, \quad (I_n, J_n) \neq (0, 0). \tag{34}$$

Let us consider the system of string equations (5) at points $(x, \boldsymbol{t}_c, u, v)$ near a given critical point $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_n$. In terms of symmetric variables it reads as

$$\begin{cases} \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i} z} V_z(z, \boldsymbol{t}_c) \mathbb{V}(z) = 0, \\ \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \boldsymbol{t}_c) \mathbb{U}(z) = -2x. \end{cases} \tag{35}$$

Then if we set

$$x = x_c + \bar{\epsilon}^{2n}\bar{x}, \qquad \epsilon = \bar{\epsilon}^{2n+1},$$

and assume (25) we obtain a recursive method for determining the coefficients of w and v. Indeed (35) is equivalent to

$$\begin{cases} \oint_\gamma \dfrac{\mathrm{d}z}{2\pi \mathrm{i}z} V_z(z, \boldsymbol{t}_c) \overline{V}^{(2j)}(z) = 0, \\ \oint_\gamma \dfrac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \boldsymbol{t}_c) \overline{U}^{(2j)}(z) = -2x_c\delta_{j0} - 2\delta_{jn}\bar{x}. \end{cases} \tag{36}$$

For $0 \leqslant j \leqslant n - 1$ these equations are identically satisfied because of (34). For $j = n$ we get from (26) and (27) that equations (36) reduce to

$$\begin{cases} J_n\beta_n^{(2n)} + 4v_c I_n\alpha_n^{(2n)} = 0, \\ I_n\beta_n^{(2n)} + J_n\alpha_n^{(2n)} = -2\bar{x}, \end{cases} \tag{37}$$

or, equivalently, in terms of the Gel'fand–Dikii polynomials (28) we obtain a pair of decoupled ordinary differential equations for $u_\pm := v^{(2)} \pm \sqrt{v_c}u^{(2)}$

$$(J_n \pm 2\sqrt{v_c}I_n)G_{n,\pm}(u_\pm) = \mp\frac{2}{\sqrt{v_c}}\bar{x}. \tag{38}$$

Thus if the condition

$$J_n \pm 2\sqrt{v_c}I_n \neq 0,$$

is satisfied, we have the so-called *doubling property* arising in the one-cut case of the Hermitian matrix model [10, 11].

The first few cases of (38) are

$$\left(f_{uvv}^{(c)} \pm \sqrt{v_c}f_{vvv}^{(c)}\right)\left(v_c u_{\pm,\overline{xx}} + 3u_\pm^2\right) = 12\bar{x}, \qquad n = 2;$$

$$\left(f_{uvvv}^{(c)} \pm \sqrt{v_c}f_{vvvv}^{(c)}\right)\left(v_c^2 u_{\pm,\overline{xxxx}} + 10v_c u_{\pm,\overline{xx}}u_\pm + 5v_c u_{\pm,\overline{x}}^2 + 10u_\pm^3\right) = 120\bar{x}, \qquad n = 3;$$

$$\left(f_{uvvvv}^{(c)} \pm \sqrt{v_c}f_{vvvvv}^{(c)}\right)\left(v_c^3 u_{\pm,\overline{xxxxxx}} + 14v_c^2 u_\pm u_{\pm,\overline{xxxx}} + 28v_c^2 u_{\pm,\overline{xxx}}u_{\pm,\overline{x}} + 21v_c^2 u_{\pm,\overline{xx}}^2 \right.$$
$$\left. + 70v_c u_\pm^2 u_{\pm,\overline{xx}} + 70v_c u_\pm u_{\pm,\overline{x}}^2 + 35u_\pm^4\right) = 1680\bar{x}, \qquad n = 4.$$

Finally for $j > n$ system (36) yields the pair of equations

$$\sum_{l=n}^{j}\left(J_l\gamma_l^{(2j)} + I_l\eta_l^{(2j)}\right) = 0, \qquad \sum_{l=n}^{j}\left(J_l\alpha_l^{(2j)} + I_l\beta_l^{(2j)}\right) = 0, \tag{39}$$

which determine each pair $(w^{(2(j-n+1))}, v^{(2(j-n+1))})$ recursively.

4. Further critical points and regularized expansions

Let us go back to system (11) for critical points of the hodograph equations (9). It is equivalent to

$$f_v = 0, \qquad f_u = 2x, \qquad f_{uv} = \sigma\sqrt{v}f_{vv}, \qquad \sigma = \pm 1. \tag{40}$$

Let us consider solutions of the hodograph equations near critical points and assume that

$$f_v(\boldsymbol{t}_c, u, v_c) \not\equiv 0$$

(similar results are obtained by interchanging the roles of u and v). Then there exists an integer $l \geqslant 1$ verifying

$$\left(\partial_u^k f_v\right)^{(c)} = 0 \qquad (0 \leqslant k < l), \qquad \left(\partial_u^l f_v\right)^{(c)} \neq 0. \tag{41}$$

As a consequence the first hodograph equation $f_v(\boldsymbol{t}_c, u, v) = 0$ can be used to eliminate the variable u as a function of v near (u_c, v_c). Indeed, from Weierstrass' preparation theorem it follows that near (u_c, v_c) there exists a factorization

$$f_v(\boldsymbol{t}_c, u, v) = (A_0(\boldsymbol{t}_c, v) + \cdots + A_{l-1}(\boldsymbol{t}_c, v)u^{l-1} + u^l)g(\boldsymbol{t}_c, u, v), \tag{42}$$

where $A_j(0 \leqslant j < l)$ are analytic functions of v, and g is an analytic function of (u, v) which does not vanish near (u_c, v_c). Hence the solutions of the hodograph equation $f_v(\boldsymbol{t}_c, u, v) = 0$ near (u_c, v_c) are given by the roots of the polynomial factor in (42)

$$A_0(\boldsymbol{t}_c, v) + \cdots + A_{l-1}(\boldsymbol{t}_c, v)u^{l-1} + u^l = 0, \tag{43}$$

and consequently they are characterized by a Puiseux series

$$u(\boldsymbol{t}_c, v) = u_c + \sum_{k \geqslant 1} a_k(\boldsymbol{t}_c)(v - v_c)^{\frac{k}{m}}, \tag{44}$$

for a certain integer $1 \leqslant m \leqslant l$. If we now introduce the function

$$H(\boldsymbol{t}_c, w) := f_u(\boldsymbol{t}_c, u(\boldsymbol{t}_c, v_c + w^m), v_c + w^m) \qquad (w := (v - v_c)^{\frac{1}{m}}) \tag{45}$$

then at $\boldsymbol{t} = \boldsymbol{t}_c$ the second hodograph equation in (9) reads $H(\boldsymbol{t}_c, w) = 2x$.

Furthermore, it is easy to see that as a consequence of system (40) we have

$$H_w^{(c)} = f_{uu}^{(c)} u_w^{(c)} + f_{uv}^{(c)} v_w^{(c)} = \sigma \sqrt{v_c} f_{vv}^{(c)} \left(\sigma \sqrt{v_c} u_w^{(c)} + v_w^{(c)}\right).$$

On the other hand, by differentiating the identity $f_v(\boldsymbol{t}_c, u(\boldsymbol{t}_c, v_c + w^m), v_c + w^m) \equiv 0$ we get

$$f_{vu}^{(c)} u_w^{(c)} + f_{vv}^{(c)} v_w^{(c)} = f_{vv}^{(c)} \left(\sigma \sqrt{v_c} u_w^{(c)} + v_w^{(c)}\right) = 0.$$

Hence, we deduce that

$$H_w^{(c)} = 0. \tag{46}$$

In this way if we assume that there exists an integer $n \geqslant 2$ such that

$$\left(\partial_w^k H\right)^{(c)} = 0, \qquad (1 \leqslant k < n), \qquad \left(\partial_w^n H\right)^{(c)} \neq 0, \tag{47}$$

then the hodograph equation $H(\boldsymbol{t}_c, w) = 2x$ determines w as a function of x with a branch point of order $n - 1$ at $x = x_c$

Definition 2. *We will denote by* $\mathcal{C}_{lmn}$, $(l \geqslant 1, 1 \leqslant m \leqslant l, n \geqslant 2)$ *the set of critical points of* (9) *characterized by* (40), (41) *and such that*

 (i) $u = u(\boldsymbol{t}_c, v)$ *has branching order* $m - 1$ *at* $v = v_c$.
 (ii) *The corresponding function* $H(\boldsymbol{t}_c, w)$ *defined by* (45) *satisfies* (47).

Example

If we set $t_n = 0$ for $n \neq 1, 3$ then system (40) reads

$$t_{c1} + 3t_{c3}\left(u_c^2 + 2v_c\right) = 0, \qquad 6t_{c3}u_c v_c + x_c = 0, \qquad u_c = \sigma\sqrt{v_c}, \qquad \sigma = \pm 1. \tag{48}$$

Let us consider the critical points $(x_c, t_{c1}, t_{c3}, u_c, v_c)$ with $t_{c3} \neq 0$. They are given by

$$v_c = -\frac{t_{c1}}{9t_{c3}}, \qquad u_c = \sigma\sqrt{v_c},$$

where (x_c, t_{c1}, t_{c3}) are constrained by the equation

$$3x_c = 2\sigma t_{c1} \left(-\frac{t_{c1}}{9t_{c3}}\right)^{1/2}.$$

As we assume that $v_c \neq 0$, we have that $t_{c1} \neq 0$ and $u_c \neq 0$ so that (41) is satisfied by $l = 1$. Moreover the hodograph equation

$$t_{c1} + 3t_{c3}(u^2 + 2v) = 0, \tag{49}$$

leads to

$$u(\boldsymbol{t}_c, w) = \left(-2\left(w - \frac{v_c}{2}\right)\right)^{1/2}, \qquad w := v - v_c, \qquad v_c = -\frac{t_{c1}}{9t_{c3}} \neq 0.$$

Furthermore $H(\boldsymbol{t}_c, w) := -12t_{c3}u(\boldsymbol{t}_c, w)(w + v_c)$ verifies $H_{ww}(\boldsymbol{t}_c, 0) \neq 0$. Therefore, it follows that $(x_c, t_{c1}, t_{c3}, u_c, v_c) \in \mathcal{C}_{112}$.

The following statements will be useful for the subsequent discussion. They are easily proved by differentiating (45) and the identity $f_v(\boldsymbol{t}, u(w), v_c + w^m) = 0$.

Lemma 2. *Given* $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_{lmn}$ *with* $v_c \neq 0$ *then*

- *If* $l = 1$ *then* $f_{vv}^{(c)}$ *does not vanish and*

$$H_{ww}^{(c)} = -\sigma v_c^{-1/2} \left(3 f_{vv}^{(c)} + 4v_c f_{vvv}^{(c)} - 4\sigma\sqrt{v_c} f_{uvv}^{(c)}\right). \tag{50}$$

- *If* $l \geqslant 2$ *then* $f_{uu}^{(c)} = f_{uv}^{(c)} = f_{vv}^{(c)} = 0$ *and*

$$H_{ww}^{(c)} = \begin{cases} v_c f_{uvv}^{(c)} \left(u_w^{(c)}\right)^2 + 2v_c f_{vvv}^{(c)} u_w^{(c)} + f_{uvv}^{(c)}, & for \quad m = 1, \\ v_c f_{uvv}^{(c)} \left(u_w^{(c)}\right)^2, & for \quad m \geqslant 2, \end{cases} \tag{51}$$

where $u_w^{(c)}$ *satisfies*

$$\begin{cases} v_c f_{vvv}^{(c)} \left(u_w^{(c)}\right)^2 + 2 f_{uvv}^{(c)} u_w^{(c)} + f_{vvv}^{(c)} = 0, & for \quad m = 1, \\ f_{vvv}^{(c)} u_w^{(c)} = 0, & for \quad m \geqslant 2. \end{cases} \tag{52}$$

As a consequence we deduce the following conditions for critical points with $n = 2$.

Proposition 1. *Given* $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_{lm2}$ *with* $v_c \neq 0$, *then it follows that*

(i) If $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_{112}$, *then*

$$3\sigma f_{vv}^{(c)} + 4\sigma v_c f_{vvv}^{(c)} - 4\sqrt{v_c} f_{uvv}^{(c)} \neq 0. \tag{53}$$

(ii) If $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_{l12}$ *with* $l \geqslant 2$ *then*

$$f_{uvv}^{(c)2} - v_c f_{vvv}^{(c)2} \neq 0. \tag{54}$$

(iii) If $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_{lm2}$ *with* $l, m \geqslant 2$ *then*

$$f_{uvv}^{(c)} \neq 0, \qquad and \qquad f_{vvv}^{(c)} = 0. \tag{55}$$

Proof. For $l = 1$ we have that $m = 1$ so that (53) is a consequence of (50). For $l \geqslant 2$ and $m = 1$, by substituting the solution of (52) into (51) we obtain condition (54).

Finally suppose that $l \geqslant 2, m \geqslant 2$. Then according to (52) and (51) we have that $u_w^{(c)} f_{vvv}^{(c)} = 0$, and $H_{ww}^{(c)} = v_c \left(u_w^{(c)}\right)^2 f_{uvv}^{(c)}$. Therefore the conditions for $H_{ww}^{(c)} \neq 0$ are $f_{vvv}^{(c)} = 0$ and $f_{uvv}^{(c)} \neq 0$ which proves (55). $\qquad\square$

Let us consider the system of string equations (5) at points $(x, \boldsymbol{t}, u, v)$ near a given critical point $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_{lmn}$. In terms of symmetric variables it reads as

$$\begin{cases} \oint_\gamma \dfrac{\mathrm{d}z}{2\pi \mathrm{i} z} V_z(z, \boldsymbol{t}) \mathbb{V}(z) = 0, \\ \oint_\gamma \dfrac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \boldsymbol{t}) \mathbb{U}(z) = -2x. \end{cases} \tag{56}$$

One may introduce stretched variables not only for x but also for $\boldsymbol{t}$. The most symmetrical choice is

$$x = x_c + \bar{\epsilon}^{2n} \bar{x}; \qquad t_j = t_{c,j} + \bar{\epsilon}^{2n} \bar{t}_j, \qquad j \geqslant 1; \qquad \epsilon = \bar{\epsilon}^{2n+1}. \tag{57}$$

Note that $\epsilon \partial_{t_j} = \bar{\epsilon} \partial_{\bar{t}_j}$ so that (12), (57) and

$$u = u_c + \sum_{j \geqslant 2} \bar{\epsilon}^j \bar{u}^{(j)}, \qquad v = v_c + \sum_{j \geqslant 2} \bar{\epsilon}^j \bar{v}^{(j)}, \tag{58}$$

are consistent with the Lax equations (2).

We will next concentrate on the case $n = 2$ and will provide a recursive method for determining expansions of form (58) near critical points in $\mathcal{C}_{lm2}$. Thus we set

$$x = x_c + \bar{\epsilon}^4 \bar{x}, \qquad t_j = t_{c,j} + \bar{\epsilon}^4 \bar{t}_j, \qquad j \geqslant 1, \qquad \bar{\epsilon} = \epsilon^{1/5}.$$

From (56), equating the coefficients of order $\bar{\epsilon}^j$ one finds

- For $0 \leqslant j \leqslant 3$:

$$\oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \boldsymbol{t}_c) \frac{\overline{V}^{(j)}}{z} = 0, \qquad \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \boldsymbol{t}_c) \overline{U}^{(j)} = -2x_c \delta_{j0}, \tag{59}$$

- For $j \geqslant 4$:

$$\begin{cases} \oint_\gamma \dfrac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \boldsymbol{t}_c) \dfrac{\overline{V}^{(j)}}{z} + \oint_\gamma \dfrac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \bar{\boldsymbol{t}}) \dfrac{\overline{V}^{(j-4)}}{z} = 0, \\ \oint_\gamma \dfrac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \boldsymbol{t}_c) \overline{U}^{(j)} + \oint_\gamma \dfrac{\mathrm{d}z}{2\pi \mathrm{i}} V_z(z, \bar{\boldsymbol{t}}) \overline{U}^{(j-4)} = -2\bar{x} \delta_{j4}. \end{cases} \tag{60}$$

Let us first analyse system (59). For $j = 0$ it reduces to the hodograph equations

$$f_v^{(c)} = 0, \qquad f_u^{(c)} = 2x_c,$$

and for $j = 1$ is trivially verified since $\overline{U}^{(1)} \equiv \overline{V}^{(1)} \equiv 0$. Furthermore, it is straightforward to see that for $j = 2$ both equations in (59) reduce to

$$(\sigma \sqrt{v_c} \bar{u}^{(2)} + \bar{v}^{(2)}) f_{vv}^{(c)} = 0. \tag{61}$$

Hence to proceed further it is required to distinguish the two types of critical points corresponding to $\mathcal{C}_{112}$ and $\mathcal{C}_{lm2}$ with $l \geqslant 2$.

Case $\mathcal{C}_{112}$

In this case $f_{vv}^{(c)} \neq 0$ and (61) implies

$$\bar{u}^{(2)} = \overline{w}^{(2)} = -\frac{\sigma}{\sqrt{v_c}} \bar{v}^{(2)}.$$

For $j = 3$ both equations in (59) lead to

$$\overline{w}^{(3)} = -\frac{\sigma}{\sqrt{v_c}} \bar{v}^{(3)}.$$

By setting $j = 4$ in (60) we find that

$$\overline{w}^{(4)} = -\frac{\sigma}{\sqrt{v_c}}\left(\overline{v}^{(4)} + \frac{1}{8}\overline{v}_{\overline{xx}}^{(2)} + \frac{1}{2v_c}\overline{v}^{(2)^2}\right) - \frac{\sigma f_v(\overline{t}, u_c, v_c)}{\sqrt{v_c}f_{vv}^{(c)}}$$
$$+ \frac{1}{6f_{vv}^{(c)}}\left[f_{uvv}^{(c)} - \sigma\sqrt{v_c}f_{vvv}^{(c)}\right]\left(\overline{v}_{\overline{xx}}^{(2)} + \frac{6}{v_c}\overline{v}^{(2)^2}\right).$$

Moreover $\overline{v}^{(2)} = -\sigma\sqrt{v_c}\overline{u}^{(2)}$ must satisfy the Painlevé I equation

$$\Gamma^{(c)}\left(\overline{v}_{\overline{xx}}^{(2)} + \frac{6}{v_c}\overline{v}^{(2)^2}\right) = 6(2\overline{x} - f_u(\overline{t}, u_c, v_c) + \sigma\sqrt{v_c}f_v(\overline{t}, u_c, v_c)), \qquad (62)$$

where

$$\Gamma^{(c)} := -\sigma\frac{\sqrt{v_c}}{2}\left(3f_{vv}^{(c)} + 4v_c f_{vvv}^{(c)} - 4\sigma\sqrt{v_c}f_{uvv}^{(c)}\right).$$

Note that since $(x_c, \boldsymbol{t}_c, u_c, v_c) \in \mathcal{C}_{112}$ then proposition 1 implies $\Gamma^{(c)} \neq 0$.

To proceed further one uses induction in the recurrence system for $\overline{U}^{(j)}, \overline{V}^{(j)}$ to prove that the only coefficients in the expansions of $\overline{U}^{(j)}, \overline{V}^{(j)}$ depending on $\overline{v}^{(j)}, \overline{v}^{(j-1)}, \overline{v}^{(j-2)}, \overline{w}^{(j)}, \overline{w}^{(j-1)}$ and $\overline{w}^{(j-2)}$ are those corresponding to $l = 1$ or 2. Moreover one finds

$$\alpha_1^{(j)} = 2\overline{v}^{(j)} + A_1^{(j)},$$
$$\alpha_2^{(j)} = 2v_c\overline{v}_{\overline{xx}}^{(j-2)} + 12\overline{v}^{(2)}\overline{v}^{(j-2)} + 12v_c\overline{w}^{(2)}\overline{w}^{(j-2)} + A_2^{(j)},$$
$$\beta_1^{(j)} = 4v_c\overline{w}^{(j)} + \frac{v_c}{2}\overline{w}_{\overline{xx}}^{(j-2)} + 4\overline{v}^{(2)}\overline{w}^{(j-2)} + 4\overline{v}^{(j-2)}\overline{w}^{(2)} + B_1^{(j)},$$
$$\beta_2^{(j)} = 4v_c^2\overline{w}_{\overline{xx}}^{(j-2)} + 24v_c(\overline{v}^{(2)}\overline{w}^{(j-2)} + \overline{v}^{(j-2)}\overline{w}^{(2)}) + B_2^{(j)},$$
$$\gamma_1^{(j)} = \overline{w}^{(j)} + C_1^{(j)},$$
$$\gamma_2^{(j)} = v_c\overline{w}_{\overline{xx}}^{(j-2)} + 6\overline{v}^{(2)}\overline{w}^{(j-2)} + 6\overline{v}^{(j-2)}\overline{w}^{(2)} + C_2^{(j)},$$
$$\eta_1^{(j)} = 2\overline{v}^{(j)} + \frac{1}{4}\overline{v}_{\overline{xx}}^{(j-2)} + 2\overline{w}^{(2)}\overline{w}^{(j-2)} + D_1^{(j)},$$
$$\eta_2^{(j)} = 2v_c\overline{v}_{\overline{xx}}^{(j-2)} + 12v_c\overline{w}^{(2)}\overline{w}^{(j-2)} + 12\overline{v}^{(2)}\overline{v}^{(j-2)} + D_2^{(j)},$$

where $A_i^{(j)}, B_i^{(j)}, C_i^{(j)}$ and $D_i^{(j)}, j \geqslant 5, i = 1, 2$ are differential polynomials in $\overline{v}^{(2)}, \ldots, \overline{v}^{(j-3)}, \overline{w}^{(2)}, \ldots, \overline{w}^{(j-3)}$. Then, for $j \geqslant 5$ substituting (24) into the first equation in (60) yields expressions

$$\overline{w}^{(j)} = -\frac{\sigma}{\sqrt{v_c}}\overline{v}^{(j)} + K_j(\overline{t}, \overline{v}^{(2)}, \ldots, \overline{v}^{(j-2)}),$$

where K_j are differential polynomials in $\overline{v}^{(2)}, \ldots, \overline{v}^{(j-2)}$. Moreover, both equations in (60) imply that $\overline{v}^{(j-2)}$ must verify a second-order linear differential equation of the form

$$\Gamma^{(c)}\left(\overline{v}_{\overline{xx}}^{(j-2)} + \frac{12}{v_c}\overline{v}^{(2)}\overline{v}^{(j-2)}\right) = H_{j-2}(\overline{t}, \overline{v}^{(2)}, \ldots, \overline{v}^{(j-3)}), \qquad (63)$$

where H_{j-2} are differential polynomials in $\overline{v}^{(2)}, \ldots, \overline{v}^{(j-3)}$. In this way, we have a scheme for determining the coefficients $\overline{u}^{(j)}, \overline{w}^{(j)}$ in (14).

We note that if $\{\overline{v}^{(l)}, l \geqslant 2\}$ is a solution of (62) and (63) ($j \geqslant 5$), then as a consequence of the symmetry transformation (16)–(19) we have that $\{(-1)^l\overline{v}^{(l)}, l \geqslant 2\}$ is also a solution of these equations. Consequently, the differential polynomials $H_{2l+1}, l \geqslant 1$, on the right-hand side of (63) are odd polynomials in $\overline{v}^{(2j+1)}, 1 \leqslant j \leqslant l-1$ and their derivatives (for example $H_3(\overline{t}, \overline{v}^{(2)}) = 0$). Hence we may set $\overline{v}^{(2j+1)} \equiv 0$ for all $j \geqslant 1$. Analogously, since equations (62), (63) can be written in terms of $\{\overline{w}^{(l)}, l \geqslant 2\}$, we may set $\overline{w}^{(2j+1)} \equiv 0$ for all $j \geqslant 1$.

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Case $\mathcal{C}_{lm2}, l \geqslant 2$

In this case system (59) is trivially satisfied. Moreover, for $j = 4$ we get from (60) that

$$\begin{cases} \Delta^{(c)}\big(v_c \overline{v}_{\overline{x}\overline{x}}^{(2)} + 3\overline{v}^{(2)^2} + 3v_c \overline{w}^{(2)^2}\big) = 6\big(f_{uvv}^{(c)}(f_u(\overline{t}, u_c, v_c) - 2\overline{x}) - v_c f_{vvv}^{(c)} f_v(\overline{t}, u_c, v_c)\big), \\ \Delta^{(c)}\big(v_c \overline{w}_{\overline{x}\overline{x}}^{(2)} + 6\overline{w}^{(2)}\overline{v}^{(2)}\big) = 6\big(f_{uvv}^{(c)} f_v(\overline{t}, u_c, v_c) - f_{vvv}^{(c)}(f_u(\overline{t}, u_c, v_c) - 2\overline{x})\big), \end{cases} \tag{64}$$

where we denote

$$\Delta^{(c)} := v_c\big(f_{vvv}^{(c)}\big)^2 - \big(f_{uvv}^{(c)}\big)^2.$$

From proposition 1 we have that $\Delta^{(c)} \neq 0$ for critical points in $\mathcal{C}_{lm2}$ with $l \geqslant 2$.

In terms of the variables $u^{(\pm)} = \overline{v}^{(2)} \pm \sqrt{v_c}u^{(2)}$, system (64) decouples into the two Painlevé I equations

$$\big(f_{uvv}^{(c)} \pm \sqrt{v_c} f_{vvv}^{(c)}\big)(v_c u_{\pm, \overline{x}\overline{x}} + 3(u_\pm)^2) = 6(2\overline{x} - f_u(\overline{t}, u_c, v_c) \mp \sqrt{v_c} f_v(\overline{t}, u_c, v_c)).$$

Thus the doubling property is satisfied in this case.

In general, for $j \geqslant 5$ (60) leads to a second-order linear system for $\overline{w}^{(j-2)}, \overline{v}^{(j-2)}$ of the form:

$$\begin{aligned} 2v_c\big(\overline{v}_{\overline{x}\overline{x}}^{(j-2)} + 6\overline{w}^{(2)}\overline{w}^{(j-2)}\big) + 12\overline{v}^{(2)}\overline{v}^{(j-2)} &= M_{j-2}(\overline{t}, \overline{w}^{(2)}, \ldots, \overline{w}^{(j-3)}, \overline{v}^{(2)}, \ldots, \overline{v}^{(j-3)}), \\ v_c \overline{w}_{\overline{x}\overline{x}}^{(j-2)} + 6\overline{w}^{(2)}\overline{v}^{(j-2)} + 6\overline{v}^{(2)}\overline{w}^{(j-2)} &= N_{j-2}(\overline{t}, \overline{w}^{(2)}, \ldots, \overline{w}^{(j-3)}, \overline{v}^{(2)}, \ldots, \overline{v}^{(j-3)}), \end{aligned} \tag{65}$$

where M_{j-2} and N_{j-2} are differential polynomials in $\overline{w}^{(2)}, \ldots, \overline{w}^{(j-3)}$ and $\overline{v}^{(2)}, \ldots, \overline{v}^{(j-3)}$. Thus, we have a recursive procedure to construct the coefficients $\overline{w}^{(j)}, \overline{v}^{(j)}, j \geqslant 2$.

Again, due to symmetry (16)–(19), it is clear that if $(\overline{w}^{(j)}, \overline{v}^{(j)}), (j \geqslant 2)$ is solution of (65) then $((-1)^j \overline{w}^{(j)}, (-1)^j \overline{v}^{(j)}), (j \geqslant 2)$ is also a solution of (65). Hence for odd j the differential polynomials M_{j-2}, N_{j-2} are odd polynomials in $\overline{w}^{(2l+1)}, \overline{v}^{(2l+1)}, l \geqslant 1$ and their $\overline{x}$ derivatives. Therefore, we may set $\overline{w}^{(2l+1)} = 0, \overline{v}^{(2l+1)} = 0$ for all $l \geqslant 1$.

4.1. Critical processes in ideal Hele–Shaw flows

In view of the properties of asymptotic solutions of the KdV equation [13–15] and the NLS equation [16] it should be expected that the inner expansions provided by the double scaling method will be relevant when the solutions of the dispersionless or the dispersionful Toda hierarchies reach a point of gradient catastrophe. We next consider an application to an ideal model of Hele–Shaw flows supplied by the Toda hierarchy.

A Hele–Shaw cell is a narrow gap between two plates filled with two fluids: say oil surrounding one or several bubbles of air. In the set-up considered in [18] (see also [19, 20]) air is injected into two fixed points of a simply connected air bubble making the bubble break into two emergent bubbles. Before the break-off the oil–air interface remains free of cusp-like singularities and develops a smooth neck. The reversed evolution describes the merging of two bubbles. The analysis of [18] concludes that before the merging, the local structure of a small part of the interface containing the tips of the bubbles is described by a curve $Y = Y(X)$ which falls into universal classes characterized by two even integers $(4n, 2), n \geqslant 1$, and a finite number $2n$ of deformation parameters t_k. Assuming symmetry of the curve with respect to the X-axis, the general solution for the curve in the $(4n, 2)$ class is

$$Y(z) := \left(\frac{\sum_{k=1}^{2n}(k+1)t_{k+1}z^k}{\sqrt{(z-a)(z-b)}}\right)_{\oplus} \sqrt{(z-a)(z-b)}, \qquad X = z, \tag{66}$$

where a and b are the positions of the bubble's tips. Due to the physical assumptions of the problem, the expansion

$$Y(z) = \sum_{k=1}^{2n}(k+1)t_{k+1}z^k + \sum_{k=0}^{\infty} \frac{Y_n}{z^n}, \qquad z \to \infty, \tag{67}$$

must satisfy the conditions $Y_0 = t$ (physical time) and $Y_1 = 0$ (bubble merging condition) which determine the positions a, b of the tips. However if $Y_1 \neq 0$ the evolution process leads to a critical point in which cusp-like singularities appear.

As was shown in [18] the positions of the bubble's tips are determined by the pair of hodograph equations

$$\sum_{k=1}^{\infty} kt_k r_{k-1}(u, v) = 0, \qquad \sum_{k=1}^{\infty} kt_k r_k(u, v) + 2x = 0, \tag{68}$$

where $t_k = 0$ for $k > 2n$, $Y_1 = 2x$ and

$$r := \frac{z}{\sqrt{(z - u)^2 - 4v}} = \sum_{k \geqslant 0} \frac{r_k(u, v)}{z^k}, \qquad a := u - 2\sqrt{v}, \qquad b := u + 2\sqrt{v}. \tag{69}$$

These are precisely the hodograph equations (6). Thus the double scaling limit method can be used to regularize the solutions of (66)–(68) at critical points in terms of inner expansions of solutions of (5).

As an example let us analyse the critical process of a merging of two bubbles studied in section VII of [18]. We set $t := t_1, t_2 = 0, t_n = 0, n > 3$ and fix t_3 to a given constant value c. Thus the Hele–Shaw interface is locally characterized by the curve

$$Y(X) = 3c(X + u)\sqrt{(X - u)^2 - 4v}, \tag{70}$$

and (68) reduces to

$$t + 3c(u^2 + 2v) = 0, \qquad 6cvu + x = 0. \tag{71}$$

This is the hodograph system (48) and its solution is given by

$$v = \frac{t^2}{2^{2/3}9c\sqrt[3]{9\sqrt{81c^2x^4 + 4cx^2t^3 - 81cx^2 - 2t^3}}} - \frac{t}{18c}$$

$$+ \frac{\sqrt[3]{9\sqrt{81c^2x^4 + 4ct^3x^2 - 81cx^2 - 2t^3}}}{18\sqrt[3]{2}c}, \qquad u = -\frac{x}{6cv}. \tag{72}$$

As we have seen above the critical points $(x_c, t_c, t_{c3}, u_c, v_c)$ with $t_c \neq 0$ are in $\mathcal{C}_{112}$ so that our scheme can be applied to provide a dispersive regularization of (72) near critical points.

Setting $x = x_c$ in (72) one finds that the outer approximations for u and v near $t = t_c$ are given by

$$u \sim u_{\text{out}}(t) := u_c - \frac{1}{3}\sqrt{\frac{1}{c}(t_c - t)}, \qquad v \sim v_{\text{out}}(t) := v_c + \frac{u_c}{3}\sqrt{\frac{1}{c}(t_c - t)},$$

$$\text{as} \quad t \to t_c^-, \tag{73}$$

where $u_c = \sigma\sqrt{v_c}$. Since we consider a critical point in $\mathcal{C}_{112}$ we introduce the stretched variables

$$x = x_c + \bar{\epsilon}^4\bar{x}, \qquad t = t_c + \bar{\epsilon}^4\bar{t}$$

and consider the inner expansions for u and v

$$u \sim u_{\text{in}} := u_c + \bar{\epsilon}^2\bar{u}^{(2)}(\bar{x} - u_c\bar{t}), \quad v \sim v_{\text{in}}(\bar{t}) := v_c + \bar{\epsilon}^2\bar{v}^{(2)}(\bar{x} - u_c\bar{t}), \quad \text{as} \quad \bar{t} \to 0, \tag{74}$$

where $\bar{u}^{(2)} = -\bar{v}^{(2)}/u_c$ and $\bar{v}^{(2)}$ verifies the Painlevé I equation

$$\bar{v}^{(2)}_{\bar{x}\bar{x}} + \frac{6}{u_c^2}(\bar{v}^{(2)})^2 = \frac{2}{3cu_c}(\bar{x} - u_c\bar{t}). \tag{75}$$

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The inner approximation at $\overline{x} = 0$ must match the outer approximation in an overlap interval which has both $t - t_c$ small and $\overline{t}$ large. Writing the outer approximations (73) in terms of the inner variable $\overline{t}$ we have

$$u_{\text{out}}(t) = u_c - \overline{\epsilon}^2 \frac{1}{3} \sqrt{-\frac{\overline{t}}{c}}, \qquad v_{\text{out}}(t) = v_c + \overline{\epsilon}^2 \frac{u_c}{3} \sqrt{-\frac{\overline{t}}{c}}.$$

Hence, it is clear that matching requires a solution $\overline{v}^{(2)}(\overline{x} - u_c\overline{t})$ of (75) satisfying

$$\overline{v}^{(2)}(-u_c\overline{t}) \sim \frac{u_c}{3} \sqrt{-\frac{\overline{t}}{c}}, \qquad \text{as} \quad \overline{t} \to -\infty. \tag{76}$$

Now, if we introduce the change of variables

$$W = -\left(\frac{3c}{2u_c^2}\right)^{2/5} \overline{v}^{(2)}, \qquad \xi = \left(\frac{2}{3cu_c^3}\right)^{1/5} (\overline{x} - u_c\overline{t}),$$

it follows that W must satisfy the P-I equation

$$W_{\xi\xi} = 6W^2 - \xi,$$

and

$$W \sim -\sqrt{\frac{\xi}{6}}, \qquad \text{as} \quad \xi \to \infty,$$

so that W must be the tritronquée solution of the P-I equation [24, 25]. Using a numerical approximation of W one finds [27] that the regularized evolution is as follows: the right bubble develops a cusp, then a new bubble appears at this cusp and it grows until it merges with the tip of the left bubble. Finally, the tip of the left bubble absorbs the new one and stays joined to the right bubble.

Acknowledgments

References

[1] Martínez Alonso L and Medina E 2007 *J. Phys. A: Math. Theor.* **40** 14223
[2] Takasaki K and Takebe T 1995 *Rev. Math. Phys.* **7** 743
[3] Brezin E and Kazakov V 1990 *Phys. Lett.* B **236** 144
[4] Douglas M and Shenker S 1990 *Nucl. Phys.* B **335** 635
[5] Gross D and Migdal A 1990 *Phys. Rev. Lett.* **64** 127
[6] Di Francesco P, Ginsparg P and Zinn-Justin Z 1995 *Phys. Rep.* **254** 1
[7] Fokas A S, Its A R and Kitaev A V 1992 *Commun. Math. Phys.* **147** 395
[8] Bleher P M and Its A R 1999 *Ann. Math.* **150** 185
[9] Bleher P M and Eynard B 2003 *J. Phys. A: Math. Gen.* **36** 2085
[10] Petropoulos P M S 1990 *Phys. Lett.* B **247** 363
[11] Bachas C and Petropoulos P M S 1991 *Phys. Lett.* B **261** 402
[12] Hollowood T, Miramontes L, Pasquinucci A and Nappi C 1992 *Nucl. Phys.* B **373** 247
[13] Grava T and Klein C 2005 Numerical solution of the small dispersion limit of Korteweg de Vries and Whitham equations *Preprint* math-ph/0511011
[14] Grava T and Klein C 2007 Numerical study of a multiscale expansion of KdV and Camassa–Holm equation *Preprint* math-ph/0702038

[15] Claeys T and Grava T 2008 Universality of the break-up profile for the KdV equation in the small dispersion limit using the Riemann–Hilbert approach (*Preprint* arXiv:0801.2326)

[16] Dubrovin B, Grava T and Klein C 2007 On universality of critical behaviour in the focusing nonlinear Schõdinger equation, elliptic umbilic catastrophe and the tritronquée solution to the Painlevé-I equation *Preprint* arXiv:0704.0501

[17] Kudryashov N A and Soukharev M B 1998 *Phys. Lett.* A **237** 206

[18] Lee S-Y, Bettelheim E and Wiegmann P 2006 *Physica* D **219** 23

[19] Teodorescu R, Wiegmann P and Zabrodin A 2005 *Phys. Rev. Lett.* **95** 044502

[20] Bettelheim E, Wiegmann P, Agam O and Zabrodin A 2005 *Phys. Rev. Lett.* **95** 244504

[21] Mineev-Weinstein M, Wiegmann P and Zabrodin A 2000 *Phys. Rev. Lett.* **84** 5106

[22] Wiegmann P W and Zabrodin P B 2000 *Commun. Math. Phys.* **213** 523

[23] Krichever I, Mineev-Weinstein M, Wiegmann P and Zabrodin A 2004 *Physica* D **198** 1

[24] Boutroux P 1913 Les transcendents de Painlevé et le asymptotique des equations différentielles du 2-ordre *Ann. Ecole Norm* **30** 265

[25] Joshi N and Kitaev A 2001 On Boutroux's tritronquée solutions of the first Painlevé equation *Stud. Appl. Math.* **107** 253

[26] Adler M, van Moerbeke P and Vanhaecke P 2006 Moment matrices and multi-component KP, with applications to random matrix theory *Preprint* math-ph/06122064

[27] Martínez Alonso L and Medina E 2007 Regularization of Hele–Shaw flows, multiscaling expansions and the Painleve I equation *Preprint* arXiv:0710.3731